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Finite rank harmonic operator-valued functions

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Abstract

The purpose of this paper is to characterize when a harmonic function with values in the finite rank operators on a Hilbert space is expressible as a harmonic matrix-valued function. We show that harmonic function with values in the rank 1 normal operators is expressible as a harmonic matrix-valued function. We also prove that for any natural number, n , a harmonic function with values in the rank n non-negative operators is expressible as a matrix-valued function and we give examples showing that these decomposition theorems fail when various hypotheses are relaxed. © 2001 Elsevier Science Inc. All rights reserved.

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Throughout this paper, let \mathcal{H} be a separable Hilbert space. Denote by $\text{Kom}(\mathcal{H})$ the algebra of compact operators on \mathcal{H} and let $\text{Rank}^n(\mathcal{H}) \subseteq \text{Kom}(\mathcal{H})$ denote the rank n operators on \mathcal{H} , where an operator A is *rank n* if the range of A and the orthogonal complement of the kernel of A are both of dimension less than or equal to n .

Recall that a function v from an open subset \mathcal{U} of the complex plane to the bounded linear operators on \mathcal{H} , $\mathcal{L}(\mathcal{H})$, is called *harmonic* if the composition of v with every bounded linear functional on $\mathcal{L}(\mathcal{H})$ is a harmonic complex-valued function. For a function taking values in the compact operators, there are several simpler characterizations of when it is harmonic (cf. [1]). In particular, $v : \mathcal{U} \rightarrow \text{Kom}(\mathcal{H})$ is harmonic if and only if for each orthonormal basis $\{\phi_i\}_{i=1}^\infty$ of \mathcal{H} , the functions $\langle v(z)\phi_i, \phi_i \rangle$ are harmonic.

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We say a harmonic function $v : \mathcal{U} \rightarrow \text{Rank}^n(\mathcal{H})$ is *expressible as a harmonic $k \times k$ matrix-valued function* if there exists a set of orthonormal vectors $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ in \mathcal{H} such that for every $z \in \mathcal{U}$ the range and orthogonal complement of the kernel of the operator $v(z)$ are spanned by these k vectors. In other words, if we extend $\{\gamma_i\}_{i=1}^k$ to form an orthonormal basis $\{\gamma_i\}_{i=1}^\infty$ of \mathcal{H} , and express each $v(z)$ in this basis, then each $v(z)$ will have non-zero entries only in its upper $k \times k$ block and, as functions of z , the matrix entries of v will all be harmonic.

Those finite rank operator-valued functions which are expressible as matrix-valued functions form an important subclass in applications which involve perturbing an operator by a “smooth” finite rank operator-valued function. (See [2,4,5] for examples of finite rank perturbation problems.) The purpose of this note is to characterize when a finite rank operator-valued function is expressible as a matrix-valued function. We begin by showing that a harmonic rank n operator-valued function with values in the non-negative operators is expressible as a harmonic $k \times k$ matrix-valued function for some $k \leq n$. The same proof establishes this result for harmonic rank n operator-valued function with values in the non-positive operators.

Lemma 1. *Let \mathcal{U} be an open, connected, simply connected neighborhood in \mathbb{C} and let $v : \mathcal{U} \rightarrow \text{Rank}_\mathscr{P}^n(\mathcal{H})$ be a harmonic function taking values in the non-negative rank n operators. Then there exists a $k \leq n$ such that v can be expressed as a harmonic $k \times k$ matrix-valued functions. Moreover, the rank of each $v(z)$ is independent of z .*

Proof. The result is trivial if v is identically zero and so we assume it is not. Fix $z_0 \in \mathcal{U}$ such that $\text{Rank}(v(z_0))$ is minimal and let k denote $\text{Rank}(v(z_0))$. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be an orthonormal basis for range of $v(z_0)$. Let $\mathcal{S} = \text{Ker}(v(z_0)) = \text{Range}(v(z_0))^\perp$. We will show for each $z \in \mathcal{U}$ and every $\beta \in \mathcal{S}$, $\langle v(z)\beta, \beta \rangle = 0$. It follows that v is expressible as a harmonic $k \times k$ matrix-valued function in the vectors $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$.

Let $\beta \in \mathcal{S}$. Then $\langle v(z)\beta, \beta \rangle$ is a non-negative harmonic function on \mathcal{U} which vanishes at z_0 . The mean value property for harmonic functions implies that there exists a positive real number ρ (depending on β) such that for any fixed r in the interval $[0, \rho]$,

$$\langle v(z_0)\beta, \beta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle v(z_0 + r e^{i\theta})\beta, \beta \rangle d\theta.$$

Since $\langle v(z_0)\beta, \beta \rangle = 0$ and the integrand is a non-negative continuous function of θ , this shows that $\langle v(z_0 + r e^{i\theta})\beta, \beta \rangle = 0$ for every $r \in [0, \rho]$ and every $\theta \in [0, 2\pi]$. In other words, $\langle v(z)\beta, \beta \rangle$ is identically zero on the disk $\Delta(z_0, \rho)$. Therefore $\langle v(z)\beta, \beta \rangle$ is identically zero on \mathcal{U} , since \mathcal{U} is simply connected and $\langle v(z)\beta, \beta \rangle$ is a real-valued harmonic function.

Finally, because z_0 was chosen so that the rank of $v(z_0)$ is minimal as z varies over \mathcal{U} , we see from the above proof that the rank of $v(z)$ is necessarily k for each $z \in \mathcal{U}$. \square

One consequence of the previous lemma is that a harmonic function $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{P}}^n(\mathcal{H})$ from an open, connected, simply connected subset of the complex plane to the non-negative rank n operators necessarily has a harmonic conjugate on \mathcal{U} with values in the rank n operators. Of course, we need the assumption that \mathcal{U} is simply connected to establish the existence of a harmonic conjugate to v , however we do not actually need the hypothesis that \mathcal{U} is simply connected in Lemma 1. We show this in the next theorem.

Theorem 1. *Let \mathcal{U} be an open, connected neighborhood in \mathbb{C} and let $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{P}}^n(\mathcal{H})$ be a harmonic function taking values in the non-negative rank n operators. Then there exists a $k \leq n$ such that v can be expressed as a harmonic $k \times k$ matrix-valued functions. Moreover, the rank of each $v(z)$ is independent of z .*

Proof. Let v be as above and fix a point a in \mathcal{U} and an open disc $\Delta(a, r)$ in \mathcal{U} . We know from Lemma 1 that there is an orthonormal set of vectors $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ such that the restriction of v to this disc, $v(z)|_{\Delta(a, r)}$ is expressible as a $k \times k$ matrix-valued harmonic function. We will show that for any $w \in \mathcal{U}$, there is an open neighborhood, W , of w in \mathcal{U} on which $v(z)|_W$ can be expressed as a $k \times k$ matrix-valued harmonic function in these same vectors $\{\gamma_i\}_{i=1}^k$.

In view of Lemma 1, the existence of the neighborhood W of w will be established if we can show that there is an open connected, simply connected subset of \mathcal{U} containing both a and w . Although the proof is elementary, for completeness we show this now. Since \mathcal{U} is an open connected subset of \mathbb{C} , it is path-wise connected. Choose a finite length piece-wise smooth arc in \mathcal{U} , $A(a, w)$, which connects a and w . Then $A(a, w)$ is compact and the collection of open discs

$$\{\Delta(b, r_b) \subseteq \mathcal{U} \mid b \in A(a, w)\}$$

is an open cover of $A(a, w)$. Thus, we can choose a finite chain $\{\Delta(b_i, r_i)\}_{i=1}^n$ covering $A(a, w)$ such that $b_1 = a$, $b_n = w$, and $\Delta(b_i, r_i) \cap \Delta(b_{i+1}, r_{i+1}) \neq \emptyset$. The set $W = \bigcup_{i=1}^n \Delta(b_i, r_i)$ is an open, connected, simply connected subset of \mathcal{U} containing both a and w . \square

If we apply Theorem 1, in the case of a non-negative rank one operator-valued harmonic function, $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{P}}^1(\mathcal{H})$, we get that $\|v(z)\|$ is a harmonic function on \mathcal{U} and there exists a unit vector $\phi \in \mathcal{H}$ such that for each $z \in \mathcal{U}$

$$v(z) = \|v(z)\| \langle \cdot, \phi \rangle \phi.$$

The following example shows that this conclusion fails if we relax the “smoothness” hypothesis from harmonic to infinitely differentiable.

Example 1. Let \mathcal{U} be any open, connected set in \mathbb{C} . In this example we construct a function $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{H}}^1(\mathcal{H})$ which is C^∞ in x and y (where $z = x + iy \in \mathcal{U}$) but which is not expressible as a 1×1 matrix-valued function. Choose disjoint open subsets W_1 and W_2 of \mathcal{U} and non-empty compact subsets $K_1 \subset W_1$ and $K_2 \subset W_2$. By the C^∞ analog of Urysohn's lemma (cf. [3]) there are non-negative C^∞ functions f_1 and f_2 satisfying $f_i = 1$ on K_i and $f_i = 0$ outside W_i , for $i = 1, 2$. Define $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{H}}^1(\mathcal{H})$ by

$$v(z) = f_1(z)\langle \cdot, \phi_1 \rangle \phi_1 + f_2(z)\langle \cdot, \phi_2 \rangle \phi_2,$$

where ϕ_1 and ϕ_2 are orthogonal, non-zero vectors in \mathcal{H} .

The previous example shows that we cannot relax the notion of smoothness in Theorem 1. However, we can extend the class of operators for harmonic rank 1 operator-valued functions. We show this in the next theorem.

Theorem 2. Let \mathcal{U} be an open, connected neighborhood in \mathbb{C} and let $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{H}}^1(\mathcal{H})$ be a harmonic function taking values in the normal rank 1 operators. Then there exists a unit vector $\phi \in \mathcal{H}$ such that if $u(z) = \langle v(z)\phi, \phi \rangle$, then

$$v(z) = u(z)\langle \cdot, \phi \rangle \phi \quad \forall z \in \mathcal{U}.$$

Proof. The result is evident if $v \equiv 0$ and so we assume this is not the case. Moreover, the proof which extends Lemma 1 to Theorem 1 applies without change. Thus, we assume without loss of generality that \mathcal{U} is simply connected.

First, assume $v : \mathcal{U} \rightarrow \text{Rank}_{\mathcal{H}}^1(\mathcal{H})$ is a harmonic function taking values in the self-adjoint operators. Since v takes values in the rank 1 self-adjoint operators, at each $z \in \mathcal{U}$, $v(z)$ has a decomposition of the form

$$v(z) = \mu(z)\langle \cdot, \phi_z \rangle \phi_z$$

for some unit vector ϕ_z and some real number $\mu(z)$ (both depending on z). Let $\mathcal{U}^+ = \{z \in \mathcal{U} : \mu(z) > 0\}$ and $\mathcal{U}^- = \{z \in \mathcal{U} : \mu(z) < 0\}$. One of these sets has non-empty interior. Hence, we can fix a neighborhood \mathcal{W} in \mathcal{U} such that the restriction of v to \mathcal{W} is either a function with values in the non-positive rank 1 operators or a function with values in the non-negative rank 1 operators. In either case, Theorem 1 implies that there exists a unit vector ϕ such that

$$v(z) = \lambda(z)\langle \cdot, \phi \rangle \phi \quad \forall z \in \Delta(z_0, r),$$

where $\lambda(z) = \langle v(z)\phi, \phi \rangle$. But then $u(z) = \lambda(z)\langle \cdot, \phi \rangle \phi$ defines a harmonic function on all of \mathcal{U} which agrees with v on the neighborhood $\Delta(z_0, r)$. Since \mathcal{U} is simply connected, $v = u$ on all of \mathcal{U} . This establishes the theorem in the case where v takes values in the self-adjoint rank 1 operators.

Now, assume that v takes values in the normal rank 1 operators. The real and imaginary parts of v are harmonic self-adjoint operator-valued functions. Moreover, since v takes values in the normal operators, its real and imaginary parts inherit

the property of taking values in the rank 1 operators. Hence, we know that there exist unit vectors ϕ_1 and ϕ_2 and harmonic real-valued functions μ_1 and μ_2 such that $\operatorname{Re}(v) = \mu_1 \langle \cdot, \phi_1 \rangle \phi_1$ and $\operatorname{Im}(v) = \mu_2 \langle \cdot, \phi_2 \rangle \phi_2$. If ϕ_1 and ϕ_2 were not linearly dependent, then v would take values in the rank 2 operators, contrary to our assumption. Thus, there exists a harmonic complex-valued function λ on \mathcal{U} such that

$$v(z) = \lambda(z) \langle \cdot, \phi \rangle \phi$$

on all of \mathcal{U} . \square

In the following example, we show that if the hypothesis that v takes values in the normal operators is dropped, then v may fail to be expressible as a harmonic matrix-valued functions.

Example 2. Let \mathcal{U} be the open unit disc $\Delta(0, 1)$ in \mathbb{C} . For each $n \in \mathbb{N}$, define $\lambda_n : \mathcal{U} \rightarrow \mathbb{R}$ to be the harmonic function $\lambda_n(z) = \operatorname{Re}(z^n)$. Fix an orthonormal basis $\{\phi_i\}_{i=1}^\infty$ of \mathcal{H} and define $w : \mathcal{U} \rightarrow \mathcal{H}$ by

$$w(z) = \sum_{n=1}^{\infty} \lambda_n(z) \phi_n.$$

It is easy to check that w is a harmonic function from \mathcal{U} to \mathcal{H} . Now fix any unit vector ϕ in \mathcal{H} and define $v : \mathcal{U} \rightarrow \operatorname{Rank}^1(\mathcal{H})$ by

$$v(z) = \langle \cdot, w(z) \rangle \phi.$$

The function v is harmonic, since for any choice $\{\psi_j\}_{j=1}^\infty$ of orthonormal basis for \mathcal{H} , the diagonal matrix coefficients $\langle v(z) \psi_i, \psi_i \rangle = \langle \psi_i, w(z) \rangle \langle \phi, \psi_i \rangle$ are harmonic. Moreover, v takes values in the rank 1 operators and for each $z \in \mathcal{U}$, the range of $v(z)$ is spanned by ϕ . However, v is not expressible as a matrix-valued function. Indeed, there cannot exist vectors $\gamma_1, \dots, \gamma_k$ which span $\operatorname{Ker}(v(z))^\perp$ for all $z \in \mathcal{U}$ since if $z_1 = \frac{1}{2}$, $z_2 = \frac{1}{3}$, \dots , $z_{k+1} = \frac{1}{k+2}$, then $\operatorname{Ker}(v(z_i))^\perp$ is spanned by $w(z_i)$. The vectors $w(z_1), \dots, w(z_{k+1})$ are linearly independent and hence cannot be spanned by the set $\{\gamma_1, \dots, \gamma_k\}$. Thus, the analog of Theorem 2 fails when we drop the hypothesis that v takes values in the normal operators.

The previous example has implications for harmonic functions with values in the rank r self-adjoint operators for $r \geq 2$.

Example 3. Let \mathcal{U} , $\{\phi_n\}_{n=1}^\infty$, $\{\lambda_n(z)\}_{n=1}^\infty$ and $w(z)$ be as in the previous example. Let $\phi = \phi_1$ and let $v(z) = \langle \cdot, w(z) \rangle \phi$. Since v is harmonic, its adjoint function $v^*(z) = \langle \cdot, \phi \rangle w(z)$ is also harmonic. Hence,

$$u(z) = v(z) + v^*(z) = \langle \cdot, w(z) \rangle \phi + \langle \cdot, \phi \rangle w(z)$$

is a harmonic function with values in the rank 2, self-adjoint operators. However, there is no finite choice of vectors $\{\gamma_1, \dots, \gamma_k\}$ in \mathcal{H} such that for every $z \in \mathcal{U}$, the

function $u(z)$ can be expressed in terms of these k vectors. Indeed, as z varies over the open unit disc in \mathbb{C} , it is easy to see that the ranges of the $u(z)$ cannot all be spanned by any fixed finite collection of vectors.

Since the function u in the previous example can also be regarded as a rank r , self-adjoint operator-valued function for any r greater than or equal to 2, this shows that Theorem 1 does not generalize to finite rank self-adjoint harmonic functions.

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